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Some covariant representations of massless Fermi fields

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Abstract. We study the class of pure quasi-free gauge invariant states of the Fermion algebra. We show that the question of G-covariance of the corresponding representations (i.e. implementability of some group G of one-particle symmetry transformations) is related to a cohomology group with values in the Hilbert–Schmidt class. In four-dimensional space–time, this cohomology is shown to be non-trivial, thus leading to G-covariant non-Fock representations, when G is taken to be the space–time translation group together with rotations, or space–time translations together with boosts in one direction and rotations about it, and the Fermion mass is zero. In two space–time dimensions, some new fully Poincaré covariant representations for free massless Fermions are constructed.

1. Introduction

The C^*-algebra approach to quantum field theory (Haag and Kastler 1964) has proved useful in helping us to understand some qualitative features of quantum physics, such as the occurrence of superselection rules, spontaneously broken symmetry or the collective motion of matter. In principle, the C^*-algebra under consideration may be limited to the set of local bounded observables. In practice, it is often easier to construct a larger algebra, the so-called *field algebra*, of which the observable algebra is a subalgebra. We shall do this in this paper, by considering first the CAR-algebra over a complex Hilbert space $\mathscr{H}$, thus: Let $a:\mathscr{H}\to\mathfrak{A}$ be an antilinear map from $\mathscr{H}$ into a C^* algebra $\mathfrak{A}$ obeying the canonical anti-commutation relations (CAR)

$$a(f)a(g)+a(g)a(f)=0$$
$$a(f)a(g)^*+a(g)^*a(f)=\langle f,g\rangle_{\mathscr{H}} \qquad f,g\in\mathscr{H}. \tag{1}$$

The C^*-algebra generated by $a(\mathscr{H})$ is denoted $\mathfrak{A}(\mathscr{H})$, and is called the *CAR-algebra* (or Fermion algebra) over $\mathscr{H}$. $\mathscr{H}$ is called the *one-particle space*.

To discuss the observable algebra and its symmetries, let G be a group acting on $\mathscr{H}$ by unitary operators $\{V_\alpha, \alpha\in G\}$. Then there exists a unique $*$ automorphism of $\mathfrak{A}$, say τ_α, for each $\alpha\in G$. If V_α is a true representation, $\{\tau_\alpha, \alpha\in G\}$ realises G in Aut $\mathfrak{A}$, and τ_α reduces to V_α on the one-particle operators $a(f)$:

$$\tau_\alpha(a(f))=a(V_\alpha f) \qquad f\in\mathscr{H}. \tag{2}$$

We shall always assume that V_α is continuous in α. Then the automorphisms τ_α are continuous in α as well.

An important case is where $G = U(1)$ and $V_\alpha = e^{i\alpha}$. Then the corresponding automorphisms τ_α are called *gauge transformations*. The set of fixed points of $\{\tau_\alpha\}$ can be shown to form a sub-C^*-algebra of $\mathfrak{A}$, denoted by Q by Araki and Wyss (1963); this may be taken to be the algebra of observables of the theory.

Another important case is where $G = P$, the Poincaré group (the inhomogeneous Lorentz group), or rather, its covering group $ISL(2, \mathbb{C})$, and V is an irreducible representation of P, of mass $m \geqslant 0$ and spin (or |helicity|) $s = \frac{1}{2}, \frac{3}{2}, \ldots$. Then the C^*-algebra $\mathfrak{A}(\mathcal{H})$, together with the Poincaré automorphisms $\{\tau_L, L \in P\}$, is called the free relativistic Fermi–Dirac algebra of mass m and spin s.

We are interested in finding 'covariant' representations of $\mathfrak{A}(\mathcal{H})$; a representation π, acting on a Hilbert space H_π, is G-covariant if there exist unitary operators U_α on H_π such that

$$U_\alpha \pi(Y) U_\alpha^{-1} = \pi(\tau_\alpha(Y)) \qquad \alpha \in G \qquad Y \in \mathfrak{A}(\mathcal{H}) \tag{3}$$

where τ_α, $\alpha \in G$, is given by (2). We say π is *continuously* G-covariant if U_α (which is not unique) may be chosen to be locally continuous in α, which is equivalent to its being ray continuous. A continuously time-translation covariant representation π is said to have positive energy if the generator of time translation

$$H = -i \frac{d}{dt} U_t \Big|_{t=0}$$

may be chosen to be non-negative.

A simple way to obtain a G-covariant representation π_ω is to build H_{π_ω} from a G-invariant state ω using the Gelfand–Naimark–Segal construction (Emch 1972). In this case, continuity of U_α on H_{π_ω} follows from continuity of V_α on $\mathcal{H}$. For example, we obtain a gauge covariant representation by starting with a gauge-invariant state ω. In this paper, we shall consider gauge-invariant quasi-free states (Shale and Stinespring 1964, Segal 1962): ω_A is a state with n-point function

$$\omega_A(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m)) = \delta_{nm} \det[\langle g_i, A f_j \rangle]. \tag{4}$$

Here, A is a complex-linear operator on $\mathcal{H}$ such that $0 \leqslant A \leqslant 1$. It is known (Powers and Størmer 1970) that ω_A is pure if and only if A is a projection. We shall treat only this case. Particularly simple examples are the Fock state ω_0 ($A = 0$) and the anti-Fock state ω_1 ($A = 1$).

We could similarly obtain a P-covariant representation by starting from a P-invariant state, for example, a so-called Shale state (Shale and Stinespring 1964). However, it is known that the only representation of the free Fermi–Dirac algebra with positive energy obtained in this way (and thus having a P-invariant vacuum) is the usual Fock representation π_0 (Weinless 1969). As we are interested in representations π of the free dynamics with positive energy, we must forego the existence of a vacuum state in H_π. Instead, we shall use a state ω_A of the form (4), and shall analyse the properties that A must have in order for $\pi_{\omega_A} \equiv \pi_A$ to be covariant under some group G. This will (§ 2) give us a Hilbert–Schmidt cocycle condition, which will be both necessary and sufficient for π_A to be a continuously G-covariant representation. In § 3 we give an example of a positive-energy Poincaré covariant representation in $1+1$ dimensions, and in § 4, a four-dimensional example covariant under boosts in one direction, also with positive energy.

2. Cocycles

Let us write $B(\mathscr{H})_2$ for the set of Hilbert–Schmidt operators on a Hilbert space $\mathscr{H}$, and $B(\mathscr{H})_1$ for the set of trace-class operators. Powers and Størmer (1970) prove that a necessary and sufficient condition for two irreducible gauge-invariant quasi-free representations π_A and π_B to be unitary equivalent is that

$$A - B \in B(\mathscr{H})_2 \tag{5}$$

or that

$$A(1-B)A \in B(\mathscr{H})_1 \qquad \text{and} \qquad (1-A)B(1-A) \in B(\mathscr{H})_1. \tag{6}$$

In order for the automorphism τ_α of the form (2) to be implemented in the representation π_A, there must exist a unitary operator U_α such that

$$\pi_A(\tau_\alpha(Y)) = U_\alpha \pi_A(Y) U_\alpha^{-1} \qquad Y \in \mathfrak{A}(\mathscr{H}). \tag{7}$$

Equivalent to this is that (7) should hold for all Y of the form $a(f), f \in \mathscr{H}$. So we need the two representations of the CAR

$$\pi_A(a(f)) \qquad \text{and} \qquad \pi_A^\alpha(a(f)) \equiv \pi_A(\tau_\alpha^{-1}(a(f))) = \pi_A(a(V_\alpha^{-1}f))$$

to be equivalent. The first, π_A, contains the quasi-free state (4) as a vector state, and the second contains the quasi-free vector state ω:

$$\begin{aligned}
\omega(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m)) \\
&= \omega_A(a(V_\alpha^{-1}f_n)^* \ldots a(V_\alpha^{-1}g_m)) \\
&= \delta_{mn} \det[\langle V_\alpha^{-1}g_i, AV_\alpha^{-1}f_j\rangle] = \delta_{mn} \det[\langle g_i, V_\alpha A V_\alpha^{-1} f_j\rangle] \\
&= \omega_{V_\alpha A V_\alpha^{-1}}(a(f_n)^* \ldots a(g_m)).
\end{aligned}$$

Hence, τ_α is implemented if and only if

$$X_\alpha \equiv A - V_\alpha A V_\alpha^{-1} \in B(\mathscr{H})_2. \tag{8}$$

Suppose (8) holds for some group G. Then the map $\alpha \to X_\alpha$ is a $B(\mathscr{H})_2$-valued 1-cocycle for G, i.e. it obeys the cocycle condition

$$V_\alpha X_\beta V_\alpha^{-1} = X_{\alpha\beta} - X_\alpha. \tag{9}$$

In this context, a 1-coboundary is a map $G \to B(\mathscr{H})_2$ of the form

$$X_\alpha = C - V_\alpha C V_\alpha^{-1} \qquad C \in B(\mathscr{H})_2 \tag{10}$$

which is a particular cocycle. Two cocycles are said to be cohomologous if they differ by a coboundary. The 'first Hilbert–Schmidt cohomology group, $H^1(G, B(\mathscr{H})_2, V)$ of G relative to the representation V' is the vector space of cocycles (obeying (9)) modulo coboundaries. (The space of cocycles itself is denoted by $Z^1(G, B(\mathscr{H})_2, V)$.) We can now get our first result.

Theorem 1. Let π_A be an irreducible quasi-free gauge-invariant representation of the CAR-algebra over $\mathscr{H}$, $\tilde{G}$ a group acting on $\mathscr{H}$ by unitary operators $\{V_\alpha, \alpha \in \tilde{G}\}$, and $X_\alpha = A - V_\alpha A V_\alpha^{-1}$, $\alpha \in \tilde{G}$.

(a) The automorphism τ_α generated by V_α (equation (2)) is implemented if and only if $X_\alpha \in B(\mathscr{H})_2$. The set of such α is a subgroup G of $\tilde{G}$. π_A is G-covariant, and X_α is a cocycle for G.

(b) If $\mathscr{H}$ is separable, the cocycle X_α is continuous and π_A is continuously G-covariant.

(c) Suppose $\{V_\alpha, \alpha \in G\}$ is irreducible on $\mathscr{H}$. Then the cohomology class of the cocycle X_α is zero if and only if π_A is (unitary equivalent to) either the Fock or the anti-Fock representation, and there is a one-to-one correspondence between the set of (equivalence classes of) quasi-free gauge-invariant, G-covariant representations that are neither Fock nor anti-Fock, and the elements of $H^1(G, B(\mathscr{H})_2, V)$ given by cocycles X_α obtained from projections A (equation (8)) not having finite dimension or co-dimension.

Proof. (a) If τ_α is implemented by U_α and τ_β by U_β, so are $\tau_{\alpha^{-1}}$ (by U_α^{-1}) and $\tau_{\alpha\beta}$ (by $U_\alpha U_\beta$); thus G is a subgroup. The other statements have been proved already.

(b) Continuity of X_α means continuity in the strong topology of $B(\mathscr{H})_2$ induced by the Hilbert–Schmidt norm. X_α is the limit of a sequence of coboundaries, each continuous in α, and is therefore measurable. The continuity of X_α then follows† from a lemma (§ 6 of Araki (1970)). Denote by Ω_A the unit vector in H_{π_A} corresponding to the state ω_A. Since π_A is irreducible, the operators U_α, and therefore the unit vectors $\Omega_A^\alpha = U_\alpha \Omega_A$, $\alpha \in G$, are unique up to phase factors. We shall prove that these phase factors can be chosen such that U_α is continuous at the unit element e of G, i.e.

$$\lim_{\alpha\to e} \|U_\alpha \Phi - U_e \Phi\| = 0 \qquad \Phi \in H_{\pi_A}. \tag{11}$$

The appropriate choice of phase factors is achieved by requiring $U_e = 1$ (as usual) and

$$\langle \Omega_A, \Omega_A^\alpha \rangle = \langle \Omega_A, U_\alpha \Omega_A \rangle \geqslant 0 \qquad \alpha \text{ near } e. \tag{12}$$

It suffices to prove (11) for a dense set of vectors Φ of the form

$$\Phi = \pi_A(Y)\Omega_A \qquad Y \in \mathfrak{A}(\mathscr{H}).$$

Then

$$\begin{aligned}\|U_\alpha \Phi - \Phi\| &= \|U_\alpha \pi_A(Y)\Omega_A - U_\alpha \pi_A(Y) U_\alpha^{-1} \Omega_A + U_\alpha \pi_A(Y) U_\alpha^{-1} \Omega_A - \pi_A(Y)\Omega_A\| \\ &\leqslant \|\pi_A(Y)\| \|\Omega_A - U_\alpha^{-1} \Omega_A\| + \|\pi_A(\tau_\alpha(Y) - Y)\|.\end{aligned}$$

Since π_A is faithful and the automorphisms τ_α are continuous in α, we obtain

$$\|\pi_A(\tau_\alpha(Y) - Y)\| = \|\tau_\alpha(Y) - Y\| \xrightarrow[\alpha\to e]{} 0.$$

Thus it remains to prove that

$$\|\Omega_A - U_\alpha^{-1}\Omega_A\| = \|U_\alpha \Omega_A - \Omega_A\| = \|\Omega_A^\alpha - \Omega_A\| \xrightarrow[\alpha\to e]{} 0. \tag{13}$$

This part of the proof is trivial if ω_A is an invariant state, since then we take $U_\alpha \Omega_A = \Omega_A$.

The states of π_A corresponding to Ω_A and Ω_A^α are ω_A and $\omega_{V_\alpha A V_\alpha^{-1}}$, respectively. Define the projections

$$A' = 1 - A \qquad A_\alpha = V_\alpha A V_\alpha^{-1} \qquad A'_\alpha = 1 - A_\alpha$$

and the (non-negative) operators (in $B(\mathscr{H})_1$, by equation (6))

$$R_\alpha = A A'_\alpha A \qquad R'_\alpha = A' A_\alpha A'.$$

† We thank P Basarab-Horwath for this remark.

The distance between the two quasi-free states ω_A and ω_{A_α} in the norm of $\mathfrak{A}(\mathcal{K})^*$, the dual space of $\mathfrak{A}(\mathcal{K})$, is given by (Powers and Størmer (1970, theorem 2.6))

$$\|\omega_A - \omega_{A_\alpha}\| = 2\sqrt{1-\delta_\alpha} \qquad \delta_\alpha = \min\{\det(1-R_\alpha), \det(1-R'_\alpha)\}. \tag{14}$$

Since X_α is continuous, for any $\varepsilon > 0$

$$\|V_\alpha A V_\alpha^{-1} - A\|_2 = \|A_\alpha - A\|_2 < \varepsilon$$

in a suitable neighbourhood $N \subseteq G$ of e. ($\|\cdot\|_2$ denotes the Hilbert–Schmidt norm.) Thus for $\alpha \in N$

$$\|A_\alpha - A\|_2^2 = \mathrm{Tr}[(A_\alpha - A)^2] = \mathrm{Tr}(R_\alpha + R'_\alpha) < \varepsilon^2$$

so that

$$\mathrm{Tr}\, R_\alpha < \varepsilon^2 \qquad \mathrm{Tr}\, R'_\alpha < \varepsilon^2 < 1 \qquad \text{if } \varepsilon < 1.$$

For any non-negative operator $R \in B(\mathcal{K})_1$ with $\mathrm{Tr}\, R < 1$ (and thus $R < 1$) we have

$$\mathrm{Tr}(R^n) \leq \|R\|\, \mathrm{Tr}(R^{n-1}) \leq \mathrm{Tr}\, R \cdot \mathrm{Tr}(R^{n-1}) \leq \ldots \leq (\mathrm{Tr}\, R)^n$$

$$\begin{aligned} \mathrm{Tr}\,\lg(1-R) &= \mathrm{Tr}(-R - \tfrac{1}{2}R^2 - \tfrac{1}{3}R^3 - \ldots) \\ &\geq -\mathrm{Tr}\, R - \tfrac{1}{2}(\mathrm{Tr}\, R)^2 - \tfrac{1}{3}(\mathrm{Tr}\, R)^3 - \ldots = \lg(1 - \mathrm{Tr}\, R) \end{aligned}$$

and thus

$$\det(1-R) \equiv \exp(\mathrm{Tr}\,\lg(1-R)) \geq \exp(\lg(1-\mathrm{Tr}\, R)) = 1 - \mathrm{Tr}\, R.$$

Applying this to R_α and R'_α, we obtain

$$\delta_\alpha > 1 - \varepsilon^2 \qquad \text{if } \alpha \in N. \tag{15}$$

On the other hand, the norm distance between two vector states ω and ω', corresponding to unit vectors Ω, Ω', of an irreducible faithful representation π of a C^*-algebra $\mathfrak{A}$, is given by

$$\begin{aligned} \|\omega - \omega'\| &= \mathrm{Tr}|\,|\Omega\rangle\langle\Omega| - |\Omega'\rangle\langle\Omega'|\,| \\ &= 2(1 - |\langle\Omega, \Omega'\rangle|^2)^{1/2}. \end{aligned}$$

Comparison with (14) yields

$$|\langle\Omega_A, \Omega_A^\alpha\rangle|^2 = \delta_\alpha. \tag{16}$$

Thus, with (12), (15) and (16),

$$\begin{aligned} \|\Omega_A - \Omega_A^\alpha\|^2 &= 2 - 2\,\mathrm{Re}\,\langle\Omega_A, \Omega_A^\alpha\rangle = 2(1 - \langle\Omega_A, \Omega_A^\alpha\rangle) \\ &\leq 2(1 - \langle\Omega_A, \Omega_A^\alpha\rangle^2) = 2(1 - |\langle\Omega_A, \Omega_A^\alpha\rangle|^2) \\ &= 2(1-\delta_\alpha) < 2\varepsilon^2 \qquad \text{if } \alpha \in N \end{aligned}$$

which proves (13) and thus (11).

As π_A is irreducible, $U_\alpha U_\beta$ and $U_{\alpha\beta}$ differ at most by a phase factor, and therefore U_α induces a ray representation of G. Equation (11) implies that this ray representation is continuous near the identity, and thus everywhere on G.

If, moreover, G is a simply connected Lie group, then U_α can be chosen to be continuous everywhere (Bargmann 1954).

(c) By (5), π_A is equivalent to the Fock representation iff $\dim A < \infty$. Then $A \in B(\mathcal{K})_2$, i.e. X_α is a coboundary, thus defining the zero cohomology class. Likewise, if π_A is the anti-Fock representation, then codim $A<\infty$, $1-A\in B(\mathscr{H})_2$, thus $X_\alpha = V_\alpha(1-A)V_\alpha^{-1}-(1-A)$ is again a coboundary, and this representation, too, gives the zero cohomology class.

Conversely, if X_α has the form (8) and gives the zero cohomology class, then

$$A-V_\alpha A V_\alpha^{-1}=C-V_\alpha C V_\alpha^{-1} \qquad \text{for some } C\in B(\mathscr{H})_2.$$

Then

$$C-A=V_\alpha(C-A)V_\alpha^{-1} \qquad \text{for all } \alpha\in \mathrm{G}.$$

But V_α is irreducible, so $C-A=\lambda 1$, i.e.

$$C=A+\lambda 1=(1+\lambda)A+\lambda(1-A)\in B(\mathscr{H})_2. \tag{17}$$

This can only be true if $\lambda=0$ and dim $A<\infty$, or if $\lambda=-1$ and codim $A<\infty$. Therefore the representation is either Fock or anti-Fock. This proves the first part of (c).

Now suppose ω_A and ω_B, both of the form (4) with projections A and B, to be G-covariant states, and denote the corresponding cocycles, given by (8) and a similar formula with B, by X_α and Y_α. If $\pi_A\cong\pi_B$, then $A-B=C\in B(\mathscr{H})_2$ and $X_\alpha-Y_\alpha = C-V_\alpha C V_\alpha^{-1}$ is a coboundary. Hence equivalent representations give rise to cohomologous cocycles. Conversely, suppose that two cocycles X_α, Y_α in the same cohomology class can both be written in the form (8) with projections A and B respectively. We have to show that π_A is equivalent to π_B if X_α is not a coboundary. But $X_\alpha-Y_\alpha$ is a coboundary, so

$$\begin{aligned} X_\alpha-Y_\alpha &= A-V_\alpha A V_\alpha^{-1}-(B-V_\alpha B V_\alpha^{-1}) \\ &= C-V_\alpha C V_\alpha^{-1} \qquad C\in B(\mathscr{H})_2 \end{aligned}$$

and therefore $A-B-C$ commutes with V_α, $\alpha\in\mathrm{G}$. Since V_α is irreducible, we have

$$A-B=C+\lambda 1.$$

Since $A-B$ is Hermitean, we get $C^*-C=2\mathrm{i}\,\mathrm{Im}\,\lambda\cdot 1\in B(\mathscr{H})_2$, which is impossible unless $C^*=C$ and $\lambda\in\mathbb{R}$. If $\lambda=0$, π_A and π_B would be equivalent, as required. If $\lambda<0$, we argue as below with A and B interchanged. So we take $\lambda>0$. Then

$$A\geqslant C+\lambda 1 \qquad C\in B(\mathscr{H})_2 \qquad \lambda>0.$$

But this operator inequality is impossible unless codim $A<\infty$, which means that π_A is the anti-Fock representation, and X_α is in the zero cohomology class. (Let codim $A=\infty$, and take a countable basis $\{f_i\}$ in $(1-A)\mathscr{H}$. Since $f_i\to 0$ weakly and C is compact, $Cf_i\to 0$ strongly, and thus

$$0=\langle f_i, Af_i\rangle \geqslant \langle f_i, Cf_i\rangle+\lambda \underset{i}{\rightarrow} \lambda>0$$

which is a contradiction.) Hence, if X_α is not in the zero cohomology class, $\lambda=0$, and π_A and π_B are equivalent. This proves the theorem.

The classification of representations of $\mathfrak{A}(\mathscr{H})$ given by theorem 1 also provides a classification of representations of the observable algebra Q. This follows from the result of Araki and Wyss (1963) which implies that if π_A and π_B are inequivalent representations of $\mathfrak{A}(\mathscr{H})$, they define inequivalent representations of Q (with cyclic vectors Ω_A and Ω_B). Consequently, the states ω_A and ω_B are separated by a superselection rule.

If V_α is reducible, there exists a projection A, not 0 or 1, commuting with V_α. The representation π_A then gives rise to the zero cocycle, and is therefore covariant. This is what happens in Dirac's theory of the electron 'sea': A is the projection onto the negative energy states of $\mathscr{K}$, which is the direct sum of two subspaces, one of each sign of the energy.

Another example is the free Fermi gas, where the invariance group is $E_3 \times \mathbb{R}$, $\mathbb{R}$ being the time evolution. Here we choose A to be the projection onto states with energy below the Fermi energy. In this case, the Hamiltonian is not bounded below.

3. Poincaré covariance in 1+1 dimensions

In this section, we give examples for non-trivial cocycles in $Z^1(P_+^\uparrow, B(\mathscr{K})_2, V)$ for the irreducible representation V of zero mass and positive energy and momentum of the (proper, orthochronous) Poincaré group $P_+^\uparrow$ in two space–time dimensions. Thus, $\mathscr{K} = (\mathbb{R}^+, \mathrm{d}p/p)$ is the space of 'right-moving' waves, and the action of Poincaré transformations is

$$(V(a, \Lambda)f)(p) = \exp[\mathrm{i}(a^0 - a^1)p]f(\mathrm{e}^{-\lambda}p) \tag{18}$$

where

$$\Lambda = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \qquad a = (a^0, a^1).$$

The strategy we adopt, in this and the subsequent section, is to partition $\mathscr{K}$ according to the spectrum of the one-particle Hamiltonian H: $\mathscr{K} = \oplus_j \mathscr{K}_j$, where

$$\mathscr{K}_0 \colon H \geqslant \varepsilon_0 \qquad \mathscr{K}_j \colon \varepsilon_j \leqslant H \leqslant \varepsilon_{j-1} \qquad j = 1, 2, \ldots .$$

Suitable sequences of intervals $(\varepsilon_j, \varepsilon_{j-1})$ will be chosen later. We then have, with $F(\mathscr{K})$ and $F(\mathscr{K}_j)$ denoting the Fermion Fock space over $\mathscr{K}$ and $\mathscr{K}_j$, respectively

$$F(\mathscr{K}) = \bigotimes_j{}^{\Omega} F(\mathscr{K}_j) \qquad \Omega = \bigotimes_j \Omega_j. \tag{19}$$

Here we use the 'incomplete' infinite tensor product of von Neumann, based on the fiducial vector $\Omega = \bigotimes_j \Omega_j$ with the Fock vacua Ω_j of $F(\mathscr{K}_j)$; then Ω is the vacuum in $F(\mathscr{K})$. The direct sum $\bigoplus_j \mathscr{K}_j$ reduces the space–time translations $V(a)$; therefore, in the isomorphism (19), the Fock space translation operators $U_F(a)$ are given in terms of the Fock space operators $U_j(a)$ on $F(\mathscr{K}_j)$ by

$$U_F(a) = \bigotimes_j{}^{\Omega} U_j(a). \tag{20}$$

If in the fiducial vector the vacua Ω_j are replaced by other unit vectors $\Phi_j \in F(\mathscr{K}_j)$, (19) yields new irreducible representation spaces of the Fermion algebra. These are continuously translation covariant, with implementing operators given by a corresponding generalisation of (20), if such generalisation makes sense. The following result can then be applied.

Scholium (Kraus *et al* 1977). Suppose $V(a)$ obeys the spectral condition. Let

$$\mathscr{H}(\mathscr{K}) = \bigotimes_j{}^{\Phi} F(\mathscr{K}_j) \qquad \Phi = \bigotimes_j \Phi_j$$

where Φ_j is a Fock state in $F(\mathscr{H}_j)$. Then a sufficient condition for the infinite product

$$U(a) = \bigotimes_j^{\Phi} U_j(a)$$

to converge to a continuous unitary representation on $\mathscr{H}(\mathscr{H})$ obeying the spectral condition, is

$$\sum_j \langle \Phi_j, H_j \Phi_j \rangle < \infty \tag{21}$$

where $H_j \geqslant 0$ is the generator of time translations (the Fock Hamiltonian) on $F(\mathscr{H}_j)$. (For the proof see Kraus *et al* (1977).)

For the rest of this section, we are dealing with $\mathscr{H} = L^2(\mathbb{R}^+, dp/p)$, with V given by (18). Let $k = \lg p$; then $dp/p = dk$, and $\mathscr{H}$ is isomorphic to $L^2(\mathbb{R}, dk)$ where, by (18), the action of Λ is translation by λ, and the one-particle Hamiltonian H is multiplication by $p = e^k$. Now, let $k_j = \lg \varepsilon_j$, and take a sequence $\{f_j\}$ of normalised states in $L^2(\mathbb{R}, dk)$ of the form

$$f_j(k) = \begin{cases} C_j = \text{constant} & \text{if } k_j \leqslant k \leqslant k_{j-1} \\ 0 \quad \text{otherwise} & \end{cases} \qquad j = 1, 2, \ldots . \tag{22}$$

The normalisation condition gives

$$\|f_j\|^2 = |C_j|^2 (k_{j-1} - k_j) = 1.$$

A special case, which suffices to give us some cocycles, is obtained by chosing $k_0 = 0$ and

$$k_{j-1} - k_j = j^{1+\varepsilon} \qquad C_j = j^{-(1+\varepsilon)/2} \qquad \varepsilon > 0 \qquad j = 1, 2, \ldots . \tag{23}$$

Now we take the fiducial vector of the scholium to be

$$\Phi = \bigotimes_j \Phi_j \qquad \Phi_0 = \Omega_0 \qquad \Phi_j = a(f_j)^* \Omega_j \qquad j = 1, 2, \ldots . \tag{24}$$

i.e. except for $j = 0$, Φ_j is a one-particle state in $F(\mathscr{H}_j)$, which may be identified with $f_j \in \mathscr{H}_j$ since the one-particle subspace of $F(\mathscr{H}_j)$ is isomorphic to $\mathscr{H}_j$. Since the Fock Hamiltonians H_j act on such states as the one-particle Hamiltonian H on $\mathscr{H}$, condition (21) of the scholium is satisfied:

$$\sum_{j \geqslant 0} \langle \phi_j, H_j \phi_j \rangle = \sum_{j \geqslant 1} \langle f_j, H f_j \rangle \leqslant \sum_{j \geqslant 1} \exp(k_{j-1}) = \sum_{j \geqslant 0} \exp(k_j) < \infty \tag{25}$$

since, by (23), $k_j \leqslant k_j - k_{j-1} = -j^{1+\varepsilon}$.

The fiducial vector (24) describes a state with all the 'levels' f_i filled, i.e. the state ω_A with $A = \Sigma_i |f_i\rangle\langle f_i|$. The representation space of π_A may thus be identified with $\bigotimes_j^{\Phi} F(\mathscr{H}_j)$. Using (25) and the scholium, we see that space–time translations in π_A are implemented and obey the spectral condition.

We now show that π_A is continuously Lorentz covariant. The vectors $g_{j\lambda} = f_j - V(\Lambda) f_j$ satisfy

$$\|g_{j\lambda}\|^2 = 2|\lambda| C_j^2 = 2|\lambda| j^{-(1+\varepsilon)}. \tag{26}$$

Let $V(\Lambda) f_j = f_{j\lambda}$. Then

$$|f_j\rangle\langle f_j| - |f_{j\lambda}\rangle\langle f_{j\lambda}| = |g_{j\lambda}\rangle\langle f_j| + |f_{j\lambda}\rangle\langle g_{j\lambda}|.$$

The intervals $k_{j-1} - k_j = j^{1+\varepsilon}$ increase in size with increasing j, whereas $V(\Lambda)$ is a fixed translation $k \to k + \lambda$. So, for all j beyond a certain j_0 (depending on λ), the only values of k for which operators like

$$(|g_{j\lambda}\rangle\langle f_j|)(|g_{k\lambda}\rangle\langle f_k|) = \langle f_j, g_{k\lambda}\rangle |g_{j\lambda}\rangle\langle f_k| \tag{27}$$

can be non-zero, are $k = j, j \pm 1$. Hence, with $A_\Lambda = V(\Lambda)AV^*(\Lambda)$,

$$\begin{aligned}(A - A_\Lambda)^2 &= \Big(\sum_j (|f_j\rangle\langle f_j| - |f_{j\lambda}\rangle\langle f_{j\lambda}|)\Big)^2 \\ &= \Big(\sum_{j\le j_0} (|f_j\rangle\langle f_j| - |f_{j\lambda}\rangle\langle f_{j\lambda}|)\Big)^2 \\ &\quad + \sum_{j>j_0} \sum_{k=j,j\pm1} (|g_{j\lambda}\rangle\langle f_j| + |f_{j\lambda}\rangle\langle g_{j\lambda}|)(|g_{k\lambda}\rangle\langle f_k| + |f_{k\lambda}\rangle\langle g_{k\lambda}|).\end{aligned}$$

The trace norm of the first term is finite, say C. The second term contains for each j a k sum over ≤ 12 operators of rank 1, of the form (27) etc. The operator (27) has the trace norm

$$\|f_k\|\|g_{j\lambda}\||\langle f_j, g_{k\lambda}\rangle| \le \|g_{j\lambda}\|\|g_{k\lambda}\| \qquad \text{as } \|f_j\| = 1.$$

Similarly, since $\|f_{j\lambda}\| = 1$ as well, all 12 operators have trace norms bounded by $\|g_{j\lambda}\|^2$ or $\|g_{j\lambda}\|\|g_{j\pm1,\lambda}\|$. By (26), these bounds are never greater than $2|\lambda|(j-1)^{-(1+\varepsilon)}$, and therefore

$$\begin{aligned}\|A - A_\Lambda\|_2^2 &= \mathrm{Tr}((A - A_\Lambda)^2) \\ &\le C + \sum_{j>j_0} \sum_{k=j,j\pm1} 4\|g_{j\lambda}\|\|g_{k\lambda}\| \\ &\le C + 12\cdot 2|\lambda| \sum_{j\ge j_0} j^{-(1+\varepsilon)} < \infty.\end{aligned}$$

For small enough $|\lambda|$ (e.g. $|\lambda| < 1$), we have $j_0 = 0$, i.e. $C = 0$, and thus $\|A - A_\Lambda\|_2 \to 0$ as $\lambda \to 0$. Together with the cocycle identity (9), this shows that $A - A_\Lambda$, $\Lambda \in L$, is a continuous cocycle in $B(\mathscr{H})_2$, and π_A is indeed continuously $L_+^\uparrow$ covariant. Collecting, we obtain the following theorem.

Theorem 2. Let $A = \Sigma_j |f_j\rangle\langle f_j|$ with f_j given by (22) and (23). Then π_A is a continuously $P_+^\uparrow$-covariant non-Fock representation of the CAR with positive energy.

We finally remark that $\Sigma_j g_{j\lambda}$ converges in $\mathscr{H}$, as

$$\begin{aligned}\Big\|\sum_{j>j_0} g_{j\lambda}\Big\|^2 &\le \sum_{j,k>j_0} |\langle g_{j\lambda}, g_{k\lambda}\rangle| \\ &\le \sum_{j>j_0} \sum_{k=j,j\pm1} \|g_{j\lambda}\|\|g_{k\lambda}\| \\ &\le 3 \sum_{j\ge j_0} \|g_{j\lambda}\|^2 < \infty.\end{aligned}$$

Putting $\Sigma_j g_{j\lambda} = g(\Lambda)$, we see that $g(\Lambda)$ is a cocycle in $Z^1(L_+^\uparrow, \mathscr{H}, V)$ of the type discovered by Basarab-Horwath *et al* (1979) to give a covariant non-Fock representation of the CCR.

4. Boost-covariant representations in (3+1) dimensions

We have not been able to find any cocycles for $P_+^\uparrow$ in $s+1$ dimensions, $s>1$. This may be related to the fact that the one-particle cohomology is trivial† (Basarab-Horwath *et al* 1979). In this section, we shall give examples of cocycles for interesting subgroups. Fully covariant, but reducible, representations can then be obtained by an inducing construction (Basarab-Horwath *et al* 1979).

It is easy to construct representations which are covariant under space rotations SO(3)—or rather, the covering group SU(2)—and space–time translations. Let $\mathscr{H} = L^2(\mathbb{R}^3, \mathrm{d}^3\boldsymbol{p}/|\boldsymbol{p}|)$ (with $\boldsymbol{p}$ = momentum), and let V be the massless representation of helicity s. Naturally, we choose $s = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ to conform to the spin-statistics theorem (Streater and Wightman 1978). Now choose a decreasing sequence $\{\varepsilon_j\}$, and form the (rotation-invariant) spaces

$$\mathscr{H}_j = L^2\{\boldsymbol{p}\colon \varepsilon_{j+1} \leqslant |\boldsymbol{p}| \leqslant \varepsilon_j\} \subset \mathscr{H}.$$

In each space $\mathscr{H}_j$, we select a projection P_j, of finite dimension d_j, onto some subspace invariant under SU(2), and take $A = \Sigma_j P_j$. Choose ε_j and d_j such that $\Sigma_j d_j\varepsilon_j < \infty$. Then, by the scholium, π_A is translation covariant with positive energy. It is clearly SU(2) covariant. Similar models have been discussed by Doplicher (1966).

We now construct representations π_A covariant under boosts in one direction, and rotations about it. Standard Wigner–Mackey theory (Mackey 1963, 1976) shows that the action of a rotation by Θ about x_3 is

$$(V(\Theta)f)(\boldsymbol{p}) = \mathrm{e}^{\mathrm{i}s\Theta} f(p_1\cos\Theta + p_2\sin\Theta, -p_1\sin\Theta + p_2\cos\Theta, p_3)$$

and that the action of a boost of rapidity λ, along x_j, is

$$(V(\lambda)f)(\boldsymbol{p}) = f(\boldsymbol{p}') \qquad \boldsymbol{p}' = (p_1, p_2, -|\boldsymbol{p}|\sinh\lambda + p_3\cosh\lambda).$$

Let $A = \Sigma_j |f_j\rangle\langle f_j|$, where $f_j \in \mathscr{H}$ is a step function equal to a real constant C_j on the cylinder

$$Z_j\colon \varepsilon_{j+1} \leqslant p_3 \leqslant \varepsilon_j \qquad p_1^2 + p_2^2 \leqslant r_j^2 \tag{28}$$

and zero outside (figure 1). Since A commutes with $V(\Theta)$, rotations about x_3 are implemented in π_A. We make the choice $r_j = \varepsilon_{j+1}$, $\varepsilon_0 = 1$ and $\varepsilon_j/\varepsilon_{j+1} = \sinh(j^{1+\delta})$. Then $\varepsilon_j \to 0$ exponentially as $j \to \infty$, so the sum of the average energies of the occupied states f_j rapidly converges. By the scholium, space–time translations are implemented, and the spectrum condition holds in π_A.

Now consider boosts with $\lambda > 0$ in the third direction. They do not change p_1 or p_2, so the support of $V(\lambda)f_j$ is constrained, as for f_j, to the region $p_1^2 + p_2^2 \leqslant r_j^2$. But the support of $V(\lambda)f_j$ as a function of p_3 is shifted, so that the bottom of the cylinder Z_j is moved to the cap C_j^-, and the top is moved to the cap C_j^+. The largest shifts occur at the edges, and are measured by

$$a_j = (\varepsilon_{j+1}^2 + r_j^2)^{1/2}\sinh\lambda + \varepsilon_{j+1}\cosh\lambda = D_\lambda\varepsilon_{j+1}$$

$$b_j = (\varepsilon_j^2 + r_j^2)^{1/2}\sinh\lambda + \varepsilon_j\cosh\lambda < D_\lambda\varepsilon_j$$

with

$$D_\lambda = \sqrt{2}\sinh\lambda + \cosh\lambda \tag{29}$$

(see figure 1).

† Basarab-Horwath and Polley (1981) have now proved that the Hilbert–Schmidt cohomology is also trivial in (3 + 1) dimensions.

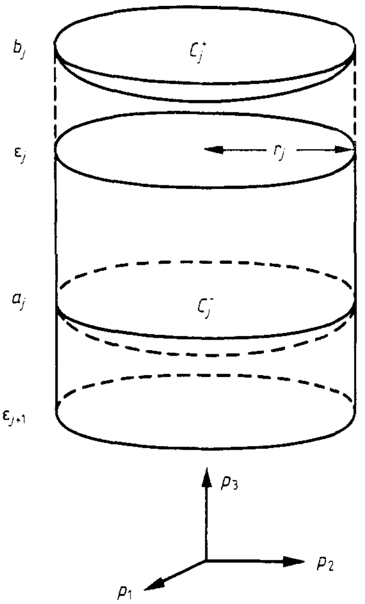

Figure 1.

For any given λ, $D_\lambda \varepsilon_{j+1} < \varepsilon_j$ for large j, and $D_\lambda \varepsilon_j < \varepsilon_{j-1}$. Hence $V(\lambda)f_j$ is orthogonal to all f_k except f_j and f_{j-1}, if j is large. The normalisation condition is, with $p = (p_1^2 + p_2^2)^{1/2}$.

$$\|f_j\|^2 = 2\pi C_j^2 \int_0^{r_j} p \,\mathrm{d}p \int_{\varepsilon_{j+1}}^{\varepsilon_j} \mathrm{d}p_3 (p_3^2 + p^2)^{-1/2} = 1$$

which leads to the inequality

$$\pi C_j^2 r_j^2 \leqslant \left(\int_{\varepsilon_{j+1}}^{\varepsilon_j} \mathrm{d}p_3 (p_3^2 + r_j^2)^{-1/2} \right)^{-1}$$

$$= [\sinh^{-1}(\varepsilon_j/\varepsilon_{j+1}) - \sinh^{-1} 1]^{-1} = (j^{1+\delta} - \sinh^{-1} 1)^{-1}. \tag{30}$$

Let $g_{j\lambda} = f_j - V(\lambda)f_j$. Then

$$\|g_{j\lambda}\|^2 \leqslant 2\pi C_j^2 \int_0^{r_j} p \,\mathrm{d}p \left(\int_{\varepsilon_{j+1}}^{a_j} \mathrm{d}p_3 (p_3^2 + p^2)^{-1/2} + \int_{\varepsilon_j}^{b_j} \mathrm{d}p_3 (p_3^2 + p^2)^{-1/2} \right)$$

$$\leqslant \pi C_j^2 r_j^2 \left(\int_{\varepsilon_{j+1}}^{D_\lambda \varepsilon_{j+1}} \frac{\mathrm{d}p_3}{|p_3|} + \int_{\varepsilon_j}^{D_\lambda \varepsilon_j} \frac{\mathrm{d}p_3}{|p_3|} \right)$$

$$= 2\pi \lg D_\lambda C_j^2 r_j^2 \leqslant 2 \lg D_\lambda (j^{1+\delta} - \sinh^{-1} 1)^{-1}$$

by (30). Therefore, as in the preamble to theorem 2,

$$(A - A_\lambda)^2 = \left(\sum_j (|f_j\rangle\langle f_j| - |f_{j\lambda}\rangle\langle f_{j\lambda}|) \right)^2$$

is of trace-class, and its trace goes to zero as $\lambda \to 0$ since, by (29), $D_\lambda \to 1$ for $\lambda \to 0$.

If $\lambda < 0$, we note that $\|f_j - V(\lambda)f_j\| = \|f_j - V(-\lambda)f_j\|$, and that $V(-\lambda)f_j$ is orthogonal to all f_k except f_j and f_{j-1} if j is large; so the calculation is the same as for $\lambda > 0$. Thus, we have the theorem given below.

Theorem 3. Let $A = \Sigma_j |f_j\rangle\langle f_j|$, where f_j is a one-particle wavefunction in $(3+1)$-dimensional space–time which is constant on its support (28) in momentum space. Then π_A is covariant under space–time translations, rotations about x_3 and boosts along x_3, and obeys the spectral condition.

We note, as before, that in this model

$$g(\Lambda) = \sum_j (f_j - V(\Lambda) f_j) \qquad \Lambda \in G$$

is a cocycle in $Z^1(G, \mathcal{H}, V)$ for the subgroup $G \subset L_+^\uparrow$ generated by the x_3-boosts and x_3-rotations, and thus defines a displaced Fock representation of the CCR (Basarab-Horwath *et al* 1979) covariant under G. (In this case, the 'natural' helicities would be $s = 0, \pm 1, \pm 2, \ldots$, but the choice of s was arbitrary anyway in the foregoing calculations.)

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